CSE 599: Interplay between Convex Optimization and Geometry **Winter 2018**

Homework 1

Lecturer: Yin Tat Lee *Due Date: 31 Jan 2018*

You are allowed to discuss with others. If you discuss with others, please list your collaborators for each question. In any case, you must write your own solutions. This homework is counted 25% of the course.

1. Proving Inequality (10 marks) Prove that for any $q \geq p > 0$ and any $x_i \geq 0$, we have

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^q\right)^{1/q} \geq \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{1/p}.$$

Hint: the function $f(x) = x^{q/p}$ is convex.

2. Dikin Ellipsoid (10 marks) Let K be the polytope $\{x \in \mathbb{R}^n : \left|a_i^\top x\right| \leq b_i \text{ for } i = 1, 2, \cdots, m\}$ where $b_i > 0$. Prove that the Dikin ellipsoid $\mathcal{E} = \left\{x \in \mathbb{R}^n : \sum_{i=1}^m \left(\frac{a_i^\top x}{b_i}\right)^2 \leq 1\right\}$ satisfies $\mathcal{E} \subseteq K \subseteq \sqrt{m}\mathcal{E}$.
Note: Compared to John ellipsoid, this is a much easier way to approximate a polytope by an ellipsoid.

3. Compute Convex Conjugate Compute the convex conjugate of following functions:

- (5 marks) $f(x) = |x|$ where $x \in \mathbb{R}$
- (5 marks) $f(x) = e^x$ where $x \in \mathbb{R}$
- (5 marks) $f(x) = a^\top x - b$ where $x, a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
- (15 marks)

$$f(X) = \begin{cases} -\log\det X & \text{if } X \succ 0 \\ +\infty & \text{else} \end{cases}$$

  where X are symmetric $n \times n$ matrices. (Remark: $f^*(Y)$ is a function on symmetry $n \times n$ matrices.)

4. Reverse of Problem 1 (25 marks) Let p be a logconcave distribution on $\mathbb{R}^n$. Prove for any $k \geq 1$,

$$\left(\mathbb{E}_{x\sim p}\left\|x\right\|_2^k\right)^{1/k} = O(k) \cdot \mathbb{E}_{x\sim p}\left\|x\right\|_2.$$

Hints: Localization lemma.

5. Compute Diameter (25 marks) Given a convex set $K \subset \mathbb{R}^n$ with $B_2 \subset K \subset R \cdot B_2$ where B_2 is the unit ball. Suppose we are given a separation oracle[1] of K with cost $\mathcal{T}$. Design a polynomial time algorithm that outputs a number α such that

$$\mathrm{diam}(K) \leq \alpha \leq 2\sqrt{n}\mathrm{diam}(K)$$

where $\mathrm{diam}(K) = \max_{x,y\in K}\left\|x - y\right\|_2$ and write the running time precisely in terms of $\mathcal{T}$ and n.
Hints: Step 1) Show how to compute an approximate smallest box $\{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}$ containing K.
Step 2) Show the diameter of the smallest box is between $\sqrt{n} \cdot \mathrm{diam}(K)$ and $\mathrm{diam}(K)$.
Note: It is possible to get $O(\sqrt{\frac{n}{\log n}})$ factor instead. But anything better requires super-polynomial time.

[1] Queried with a vector $y \in \mathbb{R}^n$, the oracle in time $\mathcal{T}$ either assert $y \in K$ or find a unit vector θ such that $\max_{x\in K}\theta^\top x \leq \theta^\top y$.