CSE 599: Interplay between Convex Optimization and Geometry **Winter 2018**

Lecture 19: Exponentially Convergent Trapezoidal Rule

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Disclaimer: *Please tell me any mistake you noticed.*

This lecture note is based on the survey [98]. As we discussed last lecture, the most simple way to calculate an integral is via

$$\int_0^1 f(x) = \frac{1}{N}\sum_{j=1}^{N} f(\frac{j-1/2}{N}).$$

Suppose we want to compute π via the formula $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$. Here is what we find using $I_h \stackrel{\text{def}}{=} h\sum_{j=-\infty}^{\infty} e^{-(jh)^2}$:

h	I_h^2
1	$3.14\cdots$
1/2	$3.141592653589793\cdots$
1/3	$\pi + 3.3 \times 10^{-38}$
1/4	$\pi + 3.3 \times 10^{-68}$
1/5	$\pi + 8.7 \times 10^{-107}$

Unlike the theory we discussed last time that gives $O(h^2)$ error, this gives $O(e^{-\pi^2/h^2})$ (ignored the truncation error). In this lecture, we will explain this super exponential convergence and its applications to various areas in mathematics.

19.1 Cauchy Integral Formula

Given any curve $\gamma : [a, b] \to \mathbb{C}$, we define

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

As an example, consider:

Lemma 19.1.1. *Let C be the curve e^{it} with $t = 0$ to 2π. We have $\int_C \frac{1}{z}dz = 2\pi i$.*

Proof. $\int_C \frac{1}{z}dz = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}dt = i\int_0^{2\pi} 1dt = 2\pi i.$ □

Definition 19.1.2. Define $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$. We call f is holomorphic on an open set U if $f'(z)$ exists for all $z \in U$.

A lot of functions we know are holomorphic (in some region), such as $\sqrt{z}, e^z, \frac{1}{1-z}$. However, there are smooth functions that are not holomorphic. For example, the function $f(z) = \overline{z}$ is not holomorphic because the value of $\frac{f(z)-f(z_0)}{z-z_0}$ varies depending on the direction which z_0 is approached. Basically, you can think holomorphic functions are functions where Taylor expansion in z converges in the complex plane. The following theorem shows that the radius of convergence of Taylor series is same as the distance to the boundary of the domain.

Theorem 19.1.3 (Convergence of Taylor Series). *Suppose f is holomorphic on an open set U. Suppose that $\{z : |z - z_0| \leq r\} \subset U$. Then for any $|z - z_0| < r$, we have that*

$$f(z_0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!}(z - z_0)^k.$$

The key result of holomorphic function is the following, which can be proved by Green's theorem:

Theorem 19.1.4 (Cauchy's integral theorem). *Suppose f is holomorphic on a simply connected open set U. For any γ in $\underline{U}$,*

$$\int_\gamma f(z)dz = 0.$$

Using this and the Lemma 19.1.1, we have

Theorem 19.1.5 (Cauchy's integral formula). *Suppose f is holomorphic on a simply connected open set U. For any a and any γ in $\underline{U}$ that enclose a,*

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a} dz.$$

Consequently, we have

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - a)^{n+1}} dz.$$

This is the main result we are going to use to analyze the trapezoidal rule. Here are two main applications of the Cauchy's integral formula we will use here. The first one shows the growth of derivatives of f depends on distance to the boundary.

Corollary 19.1.6 (Cauchy's estimate). *Suppose f is holomorphic on an open set U. Suppose that $\{z : |z - z_0| \leq r\} \subset U$, then we have*

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{r^n} \max_{z:|z-z_0|\leq r} |f(z)|.$$

The second one shows how to compute $f(A)$.

Corollary 19.1.7. *Given a symmetric matrix A with eigenvalues λ_i. Given a function f that is holomorphic on an simply connected open set U containing all λ_i. For any γ enclose all λ_i, we have*

$$f(A) = \frac{1}{2\pi i} \int_\gamma (zI - A)^{-1} f(z) dz.$$

Here is another interesting fact about holomorphic functions, which follows from the Cauchy's integral theorem.

Lemma 19.1.8. *Suppose f is holomorphic on a simply connected open set U. For any γ in $\underline{U}$, we have that*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \#\text{ of zeros inside } \gamma.$$

19.2 Trapezoidal rule for circle and interval

Theorem 19.2.1. *Suppose f is holomorphic on $\{|z| < r\}$ for some $r > 1$. Then, for any $N \geq 1$, we have that*

$$|I_N - I| \leq \frac{2\pi}{r^N - 1} M$$

where $I_N = \frac{2\pi}{N}\sum_{k=1}^{N} f(e^{2\pi ik/N})$, $I = \int_0^{2\pi} f(e^{i\theta})d\theta$ *and* $M = \max_{|z|<r}|f(z)|$*. Furthermore, the constant* 2π *is tight.*

Proof. First of all, we note that

$$\int_{|z|=1}\frac{f(z)}{z}dz = \int_0^{2\pi}\frac{f(e^{i\theta})}{e^{i\theta}}de^{i\theta} = i\int_0^{2\pi} f(e^{i\theta})d\theta.$$

Using Theorem 19.1.5, we have

$$I = -i\int_{|z|=1}\frac{f(z)}{z}dz = 2\pi f(0).$$

To compute I_N, Theorem 19.1.3 shows that $f(z) = \sum_{j=0}^{\infty}\frac{f^{(j)}(0)}{j!}z^j$ and hence

$$I_N = \frac{2\pi}{N}\sum_{k=1}^{N}\sum_{j=0}^{\infty}\frac{f^{(j)}(0)}{j!}e^{2\pi ijk/N}.$$

Note that

$$\sum_{k=1}^{N} e^{2\pi ijk/N} = \begin{cases} N & \text{if } j \text{ is a multiple of } N \\ 0 & \text{else}\end{cases}.$$

To see the second case, we note that $e^{2\pi ijk/N}$ is just points even spaced around the unit circle and hence their sum is 0. Hence, we have that

$$I_N = 2\pi\sum_{j=0}^{\infty}\frac{f^{(jN)}(0)}{(jN)!}.$$

Hence, we have that

$$I_N - I = 2\pi\sum_{j=1}^{\infty}\frac{f^{(jN)}(0)}{(jN)!}.$$

Using the Cauchy's estimate (Corollary 19.1.6), we have

$$|I_N - I| \le 2\pi\sum_{j=1}^{\infty} Mr^{-jN} = \frac{2\pi}{r^N-1}M.$$

□

Instead of expanding the function by polynomial, if we expand the function by Fourier series, we have the following:

Theorem 19.2.2. *Suppose that* f *is* 2π*-periodic, holomorphic and satisfies* $|f(z)| \le M$ *in the strip* $\{z : |Imz| \le a\}$ *for some* $a > 0$*. Then, for any* $N \ge 1$*, we have that*

$$\left|\frac{2\pi}{N}\sum_{k=1}^{N} f(\frac{2\pi k}{N}) - \int_0^{2\pi} f(\theta)d\theta\right| \le \frac{2\pi M}{e^{aN}-1}.$$

If we expand the function by Fourier transform, we have the following instead:

Theorem 19.2.3. *Suppose that* f *is holomorphic in the strip* $\{z : |Imz| \le a\}$ *for some* $a > 0$*. Suppose that* $f(z) \to 0$ *as* $|z| \to \infty$ *in the strip. Suppose that*

$$\int_{\infty}^{\infty}|f(x+ib)|dx \le M$$

for all $|b| < a$. Then, for any $N \geq 1$, we have that

$$\left| h \sum_{k=-\infty}^{\infty} f(kh) - \int_{-\infty}^{\infty} f(\theta) d\theta \right| \leq \frac{2M}{e^{2\pi a/h} - 1}.$$

Exercise 19.2.4. Applying this theorem on the example $\int_{-\infty}^{\infty} e^{-x^2} dx$, and get that

$$\left| h \sum_{k=-\infty}^{\infty} e^{-(kh)^2} - \sqrt{\pi} \right| = O(e^{a^2 - 2\pi a/h}) = O(e^{-\pi^2/h^2}).$$

Although many of these results are magical, they are magical for its simplicity, but not its convergence. There are many other ways to achieve similar results and that the equal spacing is not necessary to achieve such result if we can choose the weight as we discussed last lecture.

19.3 Applications

As we discussed, there are many applications for computing contour integration, such as computing derivatives, counting number of zeros, computing $f(A)$. Especially for $f(A)$, there are many applications of this in TCS (see [97] for some of it). Beyond this, there are lots of application of computing integral. For example, it can be used to compute special function, to solve integral equation $\int K(x,y)f(y)dy = f(x)$ and to compute Laplace transform and many other transform. Certainly, it is useful for solving ODE in various ways. The convergence of equal spacing also explains why it makes sense to solve PDE by discretize the space equally. Finally, I end the class with an example:

We know that to approximate $|x|$ by a polynomial on $[-1, 1]$ with ε error requires $\frac{1}{\varepsilon}$ degree. This is very inefficient. However, for rational function, we can use the formula

$$|x| = \frac{2x^2}{\pi} \int_{-\infty}^{\infty} \frac{e^x dx}{e^{2x} + x^2}.$$

Now, we can apply the trapezoidal rule and approximate it by

$$|x| \sim \frac{2hx^2}{\pi} \sum_{k=-(n-2)/4}^{(n-2)/4} \frac{e^{kh}}{e^{2kh} + x^2}.$$

One can check the total error (including the truncation error and discretization error) is $e^{-(\pi/2)\sqrt{n}}$. Therefore, one can approximate $|x|$ with exponentially small error using rational polynomial.

See [98] for more applications of the trapezoidal rule.

References

[97] Sushant Sachdeva, Nisheeth K Vishnoi, et al. Faster algorithms via approximation theory. *Foundations and Trends® in Theoretical Computer Science*, 9(2):125–210, 2014.

[98] Lloyd N Trefethen and JAC Weideman. The exponentially convergent trapezoidal rule. *SIAM Review*, 56(3):385–458, 2014.